Research Perspectives

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Editors
Karl-Olof Lindahl
Department of Mathematics
Linnaeus University
Växjö, Sweden

Luigi G. Rodino
Department of Mathematics
University of Torino
Torino, Italy

Patrik Wahlberg
Department of Mathematics
Linnaeus University
Växjö, Sweden

Torsten Lindström
Department of Mathematics
Linnaeus University
Växjö, Sweden

Joachim Toft
Department of Mathematics
Linnaeus University
Växjö, Sweden

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Part VII Nonlinear PDE

Part VIII *P*-adic Analysis

On Solvability of Tracking Problem Under Nonlinear Boundary Control

Erkeaim Seidakmat Kyzy and Akylbek Kerimbekov

Abstract In the paper a nonlinear boundary optimal control problem is investigated for thermal process described by Volterra integro-differential equation. Sufficient conditions are established for unique solvability of a nonlinear optimization problem. An algorithm is developed for constructing a complete solution of the nonlinear optimization problem.

1 Introduction

Applied problems described by integro-differential equations are often use in practice [1–3]. Optimal control problems were widely investigated for processes described by integro-differential equations in partial derivatives of parabolic or hyperbolic types when control is linearly included in the equation [4–7].

Many applied problems are usually nonlinear. The unique solvability of the tracking problem for nonlinear boundary control of a thermal process described by the Volterra integro-differential equation is investigated when the control function nonlinearly enters into boundary condition. A quadratic functional is the optimality criterion.

In the research process:

- a weak generalized solution of the boundary-value problem was constructed for control process, in which Fourier coefficients are defined as the solution of a linear inhomogeneous Volterra integral equation;
- optimality conditions are found by the maximum principle for systems with distributed parameters [3] and they contained a weak generalized solution of the adjoint boundary-value problem;
- nonlinear integral equation of optimal control was obtained with the additional condition in the form of a differential inequality with respect to the functions of a boundary source, and unique solvability of this problem is studied;

E. Seidakmat Kyzy (✉) · A. Kerimbekov
Kyrgyz Russian Slavic University, Bishkek, Kyrgyzstan

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- a sufficient condition was found for the unique solvability of the nonlinear optimization problem, and an algorithm for constructing its solution was developed in the form of a triple, consisting of an optimal control, an optimal process, and a minimum value of the functional.

Optimal control problem under consideration for Volterra operator possesses some specific properties which are not fulfilled when Volterra operator is replaced by Fredholm operator. For example,

- if solutions of both basic and adjoint boundary-value problems for the Volterra operator exist for any parameter λ, then solution of the boundary-value problems for Fredholm operator exists only for values of parameter λ varied only on a finite interval;
- boundary-value problems for the Volterra operator have the property of continuability with respect to the time variable t, whereas boundary-value problems for the Fredholm operator does not have this property, and it significantly affects the solvability of the optimal control problem.

Therefore, boundary optimal control problem under consideration for a thermal process is of theoretical and practical interest.

2 Formulation of the Optimal Control Problem: Optimality Conditions

Consider an optimization problem in which it is required to find the minimum value of the integral quadratic functional

$$J[u(t)] = \int_0^T \int_0^1 [V(t,x) - \xi(t,x)]^2 dx dt + \beta \int_0^T u^2(t) dt, \quad \beta > 0. \tag{1}$$

Here $V(t,x)$ characterizes the state of the controlled process, $\xi(t,x)$ describes the desired state of the controlled process for a given time, and $u(t)$ is the control function. Optimal control problem is in the definition of the control function $u^0(t)$, for which together with the corresponding to its solution $V^0(t,x)$ of following boundary-value problem:

$$\begin{gathered} V_t = V_{xx} + \lambda \int_0^t K(t,\tau) V(\tau,x) d\tau, \quad 0 < x < 1, \quad 0 < t \le T, \\ V(0,x) = \psi(x), \quad 0 < x < 1, \\ V_x(t,0) = 0, \quad V_x(t,1) + \alpha V(t,1) = p[t, u(t)], \quad 0 < t \le T, \end{gathered} \tag{2}$$

which it minimizes the functional (1). Here $K(t,\tau)$ is a given function defined in region $D = \{0 \le t \le T,\ \ 0 \le \tau \le T\}$ and satisfies the condition

$$\sup_{(t,\tau)\in D} |K(t,\tau)| = K_0,$$

$$\xi(t,x) \in H(Q),\ \ Q = \{0 < x < 1,\ \ 0 < t \le T\},\ \ \psi(t,x) \in H(0,1),\ \ p[t,u(t)] \in H(0,T) \tag{3}$$

are given functions and function $p[t,u(t)]$ is nonlinearly dependent on control function $u(t) \in H(0,T)$ and satisfies the condition, i.e.,

$$p_u[t,u(t)] \neq 0,\ \ \forall t \in (0,T), \tag{4}$$

λ is a parameter, T is fixed time moment, $\alpha > 0$, and $H(X)$ is a Hilbert space of quadratically summable functions defined on the set X.

Control $u^0(t)$ is called *an optimal control*, and the corresponding to its solution $V^0(t,x)$ is *an optimal process*. Note condition (4) ensures a one-to-one correspondence between elements of space $\{u(t)\}$ of controls and space $\{V(t,x)\}$ of solutions of the boundary-value problem (2).

To determine the optimal control we calculate the increment of functional (1). Since to each control $u(t) \in H(0,T)$ corresponds uniquely a unique solution $V(t,x) \in H(Q)$ of the boundary-value problem, the control $u(t) + \Delta u(t) \in H(0,T)$ corresponds to the solution of the boundary-value problem (2) of form $V(t,x) + \Delta V(t,x) \in H(Q)$, where $\Delta u(t)$ is increment, i.e., the solution of the boundary-value problem has an increment $\Delta V(t,x)$ corresponding to the increment $\Delta u(t)$. Similar to [3], by means of the direct calculations the increment of the functional can be represented in the form

$$\Delta J[u(t)] = J[u(t) + \Delta u(t)] - J[u(t)] =$$

$$= -\int_0^T \Delta\Pi[t, V(t,x), \omega(t,x), u(t)]dt + \int_0^T\int_0^1 \Delta V^2(t,x)dxdt,$$

where

$$\Delta\Pi[t,\cdot,u(t)] = \Pi[t,\cdot,u(t) + \Delta u(t)] - \Pi[t,\cdot,u(t)],$$

$$\Pi[t, V(t,x), \omega(t,x), u(t)] = p[t,u(t)]\omega(t,1) - \beta u^2(t), \tag{5}$$

and $\omega(t,x) \in H(Q)$ is the unique weak generalized solution (the corresponding control $u(t) \in H(0,T)$) of boundary-value problem of the form

$$\begin{aligned} \omega_t + \omega_{xx} + \lambda \int_t^T K(\tau,t)\omega(\tau,x)d\tau &= 2[V(t,x) - \xi(t,x)], \\ 0 < x < 1,\ 0 \leq t &< T, \\ \omega(T,x) = 0,\ 0 < x &< 1, \\ \omega_x(t,0) = 0,\ \omega_x(t,1) + \alpha\omega(t,1) = 0,\ 0 \leq t &< T. \end{aligned} \tag{6}$$

(6) is called adjoint boundary-value problem.

As

$$\int_0^T \int_0^1 \Delta V^2(t,x)dxdt$$

is nonnegative, the following relations hold:

1. If

$$\Delta J[u(t)] = J[u(t) + \Delta u(t)] - J[u(t)] \geq 0, \tag{7}$$

then function $\Pi[t, V(t,x), \omega(t,x), u(t)]$ must satisfy inequality

$$\Delta\Pi[t, \cdot, u(t)] = \Pi[t, \cdot, u(t) + \Delta u(t)] - \Pi[t, \cdot, u(t)] \leq 0. \tag{8}$$

2. If conditions (8) hold, then the functional satisfies condition (7).

On basis of these relations we obtain the maximum principle for function $\Pi[t, V(t,x), \omega(t,x), u^0(t)]$, the essence of which lies in the fact that for optimality of control it is necessary and sufficient that condition

$$\Pi[t, V(t,x), \omega(t,x), u^0(t)](=) \sup_{u \in Z} \Pi[t, V(t,x), \omega(t,x), u]$$

is satisfied almost everywhere on interval $[0,T]$, where Z is acceptable values set of function $u(t)$ for each fixed T.

Solving for function (5) the extremal problem for the unconstrained maximum we obtain necessary conditions in the form of the following relations:

$$\Pi_u[t, V(t,x), \omega(t,x), u(t)] = p_u[t, u(t)]\omega(t,1) - 2\beta u(t) = 0, \tag{9}$$

$$\Pi_{uu}[t, V(t,x), \omega(t,x), u(t)] = p_{uu}[t, u(t)]\omega(t,1) - 2\beta < 0, \tag{10}$$

which are called *optimality conditions.*

The optimality conditions contain solution $\omega(t,x)$ of the adjoint boundary-value problem, which make difficult the verification of condition (10). However, eliminating the function $\omega(t,x)$ from (10), we obtain the following optimality condition:

$$p_u[t,u(t)]\left(\frac{u}{p_u[t,u(t)]}\right)_u > 0. \tag{11}$$

Thus, we find optimal control according to conditions (9) and (11). We note that condition (11) restricts the class of given functions $\{p[t,u(t)]\}$, which essentially affects the solvability of the nonlinear optimization problem. Therefore, in the subsequent arguments it is assumed that condition (11) is satisfied for $\forall u(t) \in H(0,T)$. Then, to find the optimal control it suffices to consider only relation

$$\frac{2\beta u(t)}{p_u[t,u(t)]} = \omega(t,1). \tag{12}$$

We note that the solution of the adjoint boundary-value problem $\omega(t,x)$ can be found only after determining the function $V(t,x)$ according to (6), i.e., solution of the basic boundary-value problem.

3 Solution of the Basic Boundary-Value Problem

We consider the boundary-value problem (2), where the function $p[t,u(t)]$ satisfies conditions (4) and (11) for any control $u(t) \in H(0,T)$.

We are looking for a solution of problem (2) in the form

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t) z_n(x), \quad V_n(t) = \int_0^1 V(t,x), z_n(x)dx, \tag{13}$$

where

$$z_n(x) = \sqrt{\frac{2(\lambda_n^2+\alpha^2)}{\lambda_n^2+\alpha^2+\alpha}}\cos\lambda_n x, \quad n = 1,2,3,\ldots,$$

are the eigenfunctions of the boundary-value problem

$$z''(x) + \lambda_0^2 z(x) = 0, \quad z'(0) = 0, \quad z'(1) + \alpha z(1) = 0,$$

and the corresponding eigenvalues λ_n are defined as the positive roots of the transcendental equation $\lambda \tan\lambda = \alpha$ and satisfy the conditions

$$(n-1)\pi < \lambda_n < \frac{\pi}{2}(2n-1), \ \ \lambda_n < \lambda_{n+1}, \ \ n = 1,2,3,\ldots, \ \ \lim_{n\to\infty}\lambda_n = \infty.$$

Further, by means of the direct calculations we established that the Fourier coefficients $V_n(t)$ are defined as the solution of the linear inhomogeneous Volterra integral equation of the second type

$$V_n(t) = \int_0^t K_n(t,s)V_n(s)ds + a_n(t) \tag{14}$$

for each fixed $n = 1, 2, 3, \ldots$, where the function

$$K_n(t,s) = \int_s^t e^{-\lambda_n^2(t-\tau)} K(\tau, s)d\tau$$

is a kernel, and the function

$$a_n(t) = e^{-\lambda_n^2 t}\psi_n + \int_0^t e^{-\lambda_n^2(t-\tau)} z_n(1)p[\tau, u(\tau)]d\tau$$

is a free term of the integral equation.

Similar to [8], we find the solution of Eq. (14) by formula

$$V_n(t) = \lambda \int_0^t R_n(t,s,\lambda)a_n(s)ds + a_n(t), \tag{15}$$

where the resolvent $R_n(t, s, \lambda)$ of the kernel $K_n(t, s)$ is determined by Neumann series

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t,s), \; n = 1, 2, 3, \ldots. \tag{16}$$

The iterated kernels $K_{n,i}(t, s)$ are found by formulas

$$K_{n,i+1}(t,s) = \int_s^t K_n(t,\eta)K_{n,i}(\eta,s)d\eta, \;\; K_{n,1}(t,s) \equiv K_n(t,s), \;\; i = 1, 2, 3, \ldots,$$

and satisfy inequalities $\left|K_{n,i}(t,s)\right| \le \left(\frac{K_0}{\lambda_n^2}\right)^i \frac{(t-s)^{i-1}}{(i-1)!}$. According to the estimate

$$\left|R_n(t,s,\lambda)\right| \le \sum_{i=1}^{\infty} |\lambda|^{i-1} \left|K_{n,i}(t,s)\right| \le \sum_{i=1}^{\infty} |\lambda|^{i-1} \left(\frac{K_0}{\lambda_n^2}\right)^i \frac{(t-s)^{i-1}}{(i-1)!} \le$$
$$\le \frac{K_0}{\lambda_n^2} \sum_{i=1}^{\infty} \frac{1}{(i-1)!} \left(\frac{|\lambda| K_0(t-s)}{\lambda_n^2}\right)^{i-1} = \frac{K_0}{\lambda_n^2} e^{\frac{|\lambda|K_0(t-s)}{\lambda_n^2}}, \;\; n = 1, 2, 3, \ldots, \;\; 0 \le s \le t \le T,$$

$R_n(t, s, \lambda)$ is a continuous function for any value of the parameter λ for each $n = 1, 2, 3, \ldots$.

Taking into account (13) and (15), we find the solution of boundary-value problem (2) by formula

$$V(t,x)=\sum_{n=1}^{\infty}\Big\{\psi_n(t,\lambda)+\int_0^t S_n(t,\tau,\lambda)z_n(1)p[\tau,u(\tau)]d\tau\Big\}z_n(x), \qquad (17)$$

where

$$\psi_n(t,\lambda)=\psi_n\Big[e^{-\lambda_n^2 t}+\lambda\int_0^t R_n(t,s,\lambda)e^{-\lambda_n^2 t}ds\Big],$$

$$S_n(t,\tau,\lambda)=e^{-\lambda_n^2(t-\tau)}+\lambda\int_\tau^t R_n(t,s,\lambda)e^{-\lambda_n^2(s-\tau)}ds.$$

Further, by means of the direct calculations we have proved that the function $V(t,x)$ is an element of the space $H(Q)$ and this function is called *a weak generalized solution of the boundary-value problem* (2).

4 Solution of the Adjoint Boundary-Value Problem

We are looking for a solution of the adjoint boundary-value problem (6) in the form

$$\omega(t,x)=\sum_{n=1}^{\infty}\omega_n(t)z_n(x),\ \ \omega_n(t)=\int_0^1\omega(t,x),z_n(x)dx. \qquad (18)$$

Fourier coefficients $\omega_n(t)$ are defined as solution of linear inhomogeneous Volterra integral equation of the second type

$$\omega_n(t)=\lambda\int_t^T B_n(s,t)\omega_n(s)ds-2\int_t^T e^{-\lambda_n^2(\tau-t)}[V_n(\tau)-\xi_n(\tau)]d\tau, \qquad (19)$$

$$n=1,2,3,\ldots,$$

where

$$B_n(s,t)=\int_t^s e^{-\lambda_n^2(\tau-t)}K(s,\tau)d\tau.$$

Here $V_n(t)$ *and* $\xi_n(t)$ are the Fourier coefficients of functions $V(t,x)$ and $\xi(t,x)$.

The solution of Eq. (19) is determined by the formula

$$\omega_n(t) = -2\lambda \int_t^T L_n(s,t,\lambda)\Big(2\int_s^T e^{-\lambda_n^2(\tau-s)}[V_n(\tau)-\xi_n(\tau)]d\tau\Big)ds - \\ -2\int_t^T e^{-\lambda_n^2(\tau-t)}[V_n(\tau)-\xi_n(\tau)]d\tau, \quad (20)$$

where resolvent $L_n(s,t,\lambda)$ is a continuous function, as the sum of an absolutely convergent Neumann series of the form

$$L_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} B_{n,i}(s,t). \quad (21)$$

For iterated kernels $B_{n,i}(s,t)$

$$B_{n,i+1}(s,t) = \int_s^t B_n(s,\eta)B_{n,i}(\eta,t)d\eta, \quad B_n(s,t) \equiv B_{n,1}(s,t), \quad i = 1,2,3,\ldots,$$

estimates $\left|B_{n,i}(s,t)\right| \le \left(\frac{K_0}{\lambda_n^2}\right)^i \frac{(s-t)^{i-1}}{(i-1)!}$, $i = 1,2,3,\ldots$, are established. It ensures the convergence of the Neumann series (21) for any value of the parameter λ for each $n = 1,2,3,\ldots$.

Taking into account (15), (18), and (20) we find the solution of adjoint boundary-value problem by formula

$$\omega(t,x) = -2\sum_{n=1}^{\infty}\Big(\int_0^T\int_t^T E_n(t,\tau,\lambda)S_n(\tau,y,\lambda)d\tau z_n(1)p[y,u(y)]dy - \\ -\int_t^T E_n(t,\tau,\lambda)b_n(\tau,\lambda)d\tau\Big)z_n(x), \quad (22)$$

where

$$E_n(t,\tau,\lambda) = \lambda\int_t^\tau L_n(t,s,\lambda)e^{-\lambda_n^2(\tau-s)}ds + e^{-\lambda_n^2(\tau-t)},$$

$$b_n(\tau,\lambda) = \big(\xi_n(t) - \psi_n(\tau,\lambda)\big).$$

Taking into account the inequalities

$$|L_n(t,s,\lambda)| \le \frac{K_0}{\lambda_n^2}e^{\frac{|\lambda|K_0(s-t)}{\lambda_n^2}},$$

it is not difficult to prove that function $\omega(t, x)$ is an element of the space $H(Q)$ and this function is called *a weak generalized solution of adjoint boundary-value problem* (6).

5 Nonlinear Integral Equation of Optimal Control

In the optimality condition (12) substituting the function (22), we obtain the relation

$$\frac{\beta u(t)}{p_u[t, u(t)]} + \sum_{n=1}^{\infty} z_n(1) \int_0^T \left(\int_t^T E_n(t, \tau, \lambda) S_n(\tau, y, \lambda) d\tau \right) z_n(1) p[y, u(y)] dy =$$

$$= \sum_{n=1}^{\infty} z_n(1) \int_t^T E_n(t, \tau, \lambda) b_n(\tau, \lambda) d\tau, \quad (23)$$

where only control function $u(t)$ is unknown. This relation is called *the nonlinear integral equation of optimal control.*

Unique solvability of nonlinear integral equation (23) is investigated according to the procedure of work [9] tested in several studies of nonlinear optimal control problems [10–12]. Let's assume that

$$\frac{\beta u(t)}{p_u[t, u(t)]} = \nu(t). \quad (24)$$

We consider this equality as implicit function with respect to control function $u(t)$. Then, according to the optimality condition (11), Eq. (24) is uniquely resolved with respect to the function $u(t)$, i.e., there is such a function $\mu(\cdot)$ that

$$u(t) = \mu[t, \nu(t), \beta]. \quad (25)$$

According to (24)–(25), we reduce Eq. (23) to the following form:

$$\nu(t) + \sum_{n=1}^{\infty} z_n(1) \int_0^T \left(\int_t^T E_n(t, \tau, \lambda) S_n(\tau, y, \lambda) d\tau \right) z_n(1) p\big[y, \mu[y, \nu(y), \beta]\big] =$$

$$= \sum_{n=1}^{\infty} z_n(1) \int_t^T E_n(t, \tau, \lambda) b_n(\tau, \lambda) d\tau. \quad (26)$$

We introduce the notation

$$L[\nu(t)] = \sum_{n=1}^{\infty} z_n(1) \int_0^T \left(\int_t^T E_n(t,\tau,\lambda) S_n(\tau,y,\lambda) d\tau \right) z_n(1) p\big[y, \mu[y, \nu(y), \beta]\big] dy,$$

$$h = h(t,1) = \sum_{n=1}^{\infty} z_n(1) \int_t^T E_n(t,\tau,\lambda) b_n(\tau,\lambda) d\tau,$$

and we rewrite Eq. (26) in the operator form

$$\nu + L[\nu] = h. \tag{27}$$

Further, by means of the direct calculations we have proved the following lemmas:

Lemma 1 *The function $\nu(t)$ is an element of Hilbert space $H(0,T)$.*

Lemma 2 *The function $h(t,1)$ is an element of Hilbert space $H(0,T)$.*

Lemma 3 *The operator $L[\nu]$ maps spaces $H(0,T)$ into itself, i.e., it is an element of Hilbert space $H(0,T)$ for any $\nu(t) \in H(0,T)$.*

Lemma 4 *Suppose that for the function $p[t,u(t)]$ the Lipschitz condition is satisfied with respect to functional variable u, i.e.,*

$$\big|p[t,u(t)] - p[t,\overline{u}(t)]\big| \le p_0 \big|u(t) - \overline{u}(t)\big|, \quad p_0 > 0,$$

and for the function $\mu[t,\nu(t),\beta]$ it is satisfied with respect to the functional variable ν, i.e.,

$$\big|\mu[t,\nu(t),\beta] - \mu[t,\overline{\nu}(t),\beta]\big| \le \mu_0(\beta) \big|\nu(t) - \overline{\nu}(t)\big|, \quad \mu_0(\beta) > 0.$$

Then if the condition

$$\gamma = C_0 p_0 \mu_0(\beta) < 1$$

is met, operator $L[\nu]$ is a contracting operator. Here constants C_0, p_0, $\mu_0(\beta)$ are positive numbers.

Theorem 1 *Suppose that (4) and (11) and the conditions of Lemma 1–4 are satisfied. Then operator equation (27) has a unique solution in the Hilbert space $H(0,T)$.*

Proof Under the conditions of *Lemmas 1–4* the contracting mapping principle is valid, i.e., the operator $L[\nu]$ maps a complete metric space $H(0,T)$ into itself and it is the contracting operator. Therefore, by the theorem on the contracting mapping principle [13] there exists a unique fixed point for the operator $L[\nu]$, which is a solution of operator equation (27). □

Approximate solutions of operator equation (27) are constructed by the method of successive approximations

$$\nu_k = L[\nu_{k-1}] + h, \quad k = 1, 2, 3, \ldots.$$

The exact solution $\overline{\nu}(t)$ of the operator equation (27) is defined as the limit of the sequence $\{\overline{\nu}(t)\}$, i.e., $\overline{\nu}(t) = \lim_{k\to\infty} \nu_k(t)$ and it satisfies the estimate [13]

$$\|\overline{\nu}(t) - \nu_k(t)\|_{H(0,T)} \le \frac{\gamma^k}{1-\gamma}\|L[\nu_0(t)] + h(t,1) - \nu_0(t)\|_{H(0,T)},$$

where $\nu_0(t)$ is an arbitrary element of space $H(0,T)$.

Substituting the found solution $\overline{\nu}(t)$ into (27) we find the required optimal control

$$u^0(t) = \mu[t, \overline{\nu}(t), \beta], \tag{28}$$

which is a solution of the nonlinear integral equation (23).

6 Construction of the Complete Solution to the Nonlinear Optimization Problem

Substituting the optimal control (28) in (17) instead of the control $u(t)$ we obtain the optimal process

$$V^0(t,x) = \sum_{n=1}^{\infty}\left\{\psi_n(t,\lambda) + \int_0^T S_n(t,\tau,\lambda) z_n(1) p[\tau, u^0(\tau)]d\tau\right\} z_n(x),$$

i.e., the solution of the boundary-value problem (2) corresponding to the optimal control $u^0(t)$.

After determining the optimal control and the optimal process, we calculate the minimum value of the functional (1) by the formula

$$J[u^0(t)] = \int_0^T \int_0^1 [V^0(t,x) - \xi(t,x)]^2 dxdt + \beta\int_0^T [u^0(t)]^2 dt.$$

Thus, the found triple $\left(u^0(t),\ \ V^0(t,x),\ \ J[u^0(t)]\right)$ is called *a complete solution* to nonlinear optimization problem (1)–(4).

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